The Solow Growth Model: An Excel Exercise

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1 Introduction

In my Macroeconomic Analysis (ECO 62) course at Long Island University in Brookville, New York, the textbook is N. Gregory Mankiw's *Macroeconomics*, Sixth Edition, Worth Publishers, New York, New York, 2007. Although I tend to follow the text quite closely, my lectures on the Solow Growth Model use an Excel spreadsheet that relies on a slightly different treatment than the one in the textbook. This essay explains the construction of the spreadsheet.

2 The Dynamics of the Economy

Let's say we know the exact values of K_t, L_t, and E_t, which are, respectively, the economy's *capital stock*, *number of workers*, and *efficiency of labor*, at time t.

Assuming that the production function follows the Cobb-Douglas formula, we can express the economy's *total output* at time t as

$$\begin{aligned} Y_t &= F(K_t, L_tE_t) \\ &= AK_t^{\alpha}(L_tE_t)^{1-\alpha}. \end{aligned} \tag{1}$$

Assuming we know α, we can use the above formula to calculate the exact value of Y_t.[1]

Not only that, assuming we know the exact value of s, which is the economy's *saving rate*, we can also calculate sY_t, the *total saving* at time t.[2]

Let's further assume that we know the exact value of δ, the economy's *depreciation rate*.[3] Then we can figure the exact value of δK_t, which is the part of the economy's capital at time t that depreciates (or, wears out) before time $t+1$.

Once we have calculated K_t, sY_t, and δK_t, we can calculate the economy's capital stock at time $t+1$:

$$\begin{aligned} K_{t+1} &= K_t + sY_t - \delta K_t \\ &= (1-\delta)K_t + sAK_t^{\alpha}(L_tE_t)^{1-\alpha}. \end{aligned} \tag{2}$$

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[1] Note that α also happens to be the share of total output that is earned by the owners of capital; this is explained in chapter 3 of the textbook.

[2] See page 189 of the textbook for more on s.

[3] See page 191 of the textbook for more on δ.

Not just that, assuming we know n, which is the growth rate of L, and g, which is the growth rate of E, we can calculate

$$L_{t+1} = (1+n)L_t \tag{3}$$

and

$$E_{t+1} = (1+g)E_t, \tag{4}$$

which are, respectively the number of workers and the efficiency of labor at time $t+1$.[4]

Note what we have achieved so far. We started out knowing K_t, L_t, and E_t and we were then able to use that knowledge to figure out K_{t+1}, L_{t+1}, and E_{t+1}. Therefore, in the same way, we can figure out K_{t+2}, L_{t+2}, and E_{t+2}. By extension, we can figure out K, L, and E for ever and ever!

And, as we know that total output is given by equation (1), we can figure out total output (Y) for ever and ever. And once total output is known, we would be able to calculate saving (sY), consumption ($Y - sY = (1-s)Y$), per capita income (Y/L), etc., for ever and ever. In other words, we can figure out pretty much everything worth knowing about the model economy from here to eternity.

The next question is *how rapidly* will these variables *grow* in the long run? In particular, how rapidly will per capita income, on which our standard of living depends, grow in the long run?

3 Long Run Growth: Transition to the Steady State

Once we know the exact values of K_t, L_t, and E_t, we can calculate the exact values of $L_t \times E_t$ and $k_t = K_t/(L_t \times E_t)$, which are, respectively, the economy's *effective number of workers* and *capital per effective worker*.[5] Now, by dividing both sides of equation (2) by $L_{t+1}E_{t+1}$ we get

$$\begin{aligned}
k_{t+1} \equiv \frac{K_{t+1}}{L_{t+1}E_{t+1}} &= \frac{(1-\delta)K_t + sAK_t^\alpha (L_tE_t)^{1-\alpha}}{L_{t+1}E_{t+1}} \\
&= \frac{(1-\delta)K_t}{L_{t+1}E_{t+1}} + \frac{sAK_t^\alpha (L_tE_t)^{1-\alpha}}{L_{t+1}E_{t+1}} \\
&= \frac{(1-\delta)K_t}{(1+n)L_t(1+g)E_t} + \frac{sAK_t^\alpha (L_tE_t)^{1-\alpha}}{(1+n)L_t(1+g)E_t} \\
&= \frac{(1-\delta)K_t}{(1+n)L_t(1+g)E_t} + \frac{sAK_t^\alpha (L_tE_t)^{-\alpha}}{(1+n)(1+g)} \\
&= \frac{(1-\delta)}{(1+n)(1+g)}k_t + \frac{sA}{(1+n)(1+g)}k_t^\alpha.
\end{aligned} \tag{5}$$

Therefore, the *change* in capital per effective worker from time t to time $t+1$ is

$$\begin{aligned}
\Delta k_t \equiv k_{t+1} - k_t &= \left[\frac{(1-\delta)}{(1+n)(1+g)} - 1\right] k_t + \frac{sA}{(1+n)(1+g)}k_t^\alpha \\
&= \frac{(1-\delta) - (1+n)(1+g)}{(1+n)(1+g)}k_t + \frac{sA}{(1+n)(1+g)}k_t^\alpha \\
&= \frac{-(\delta+n+g+ng)}{(1+n)(1+g)}k_t + \frac{sA}{(1+n)(1+g)}k_t^\alpha \\
&= \frac{1}{(1+n)(1+g)}\left[sAk_t^\alpha - (\delta+n+g+ng)k_t\right].
\end{aligned} \tag{6}$$

[4]See page 218 of the textbook for more on n and g. The textbook refers to the latter as the rate of labor-augmenting technological progress.

[5]See page 217 of the textbook for more on these terms.

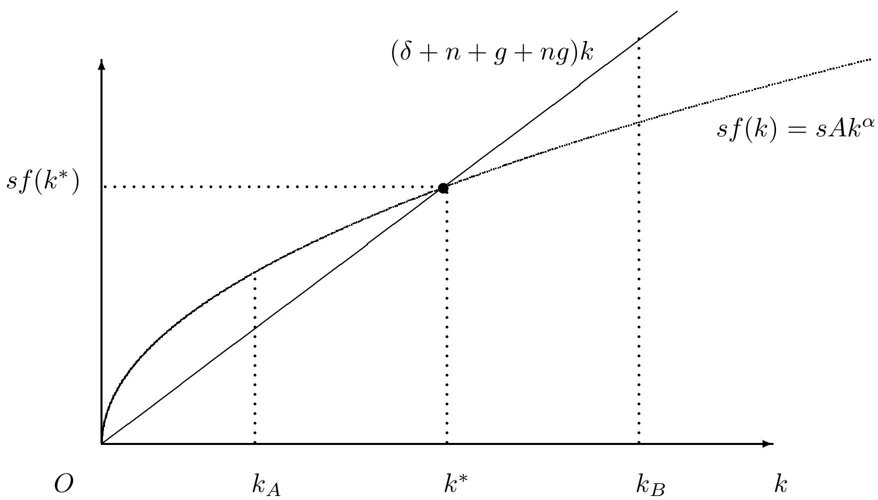

Figure 1: This graph takes a visual look at equation (8) and reveals how an economy's capital per effective worker, k, changes over time in the Solow Growth Model. The expressions sAk^α and $(\delta+n+g+ng)k$ in equation (8) can be graphed—the former becomes a concave curve and the latter becomes a line through the origin—and then visually compared to reveal to us whether the economy's initial value of k will increase or decrease over time. If $k = k_A < k^*$, then $sAk^\alpha > (\delta+n+g+ng)k$. Consequently, by equation (8), k will increase over time towards its steady state value, k^*. On the other hand, if $k = k_B > k^*$, then $sAk^\alpha < (\delta + n + g + ng)k$. Consequently, by equation (8), k will decrease over time towards k^*. In other words, no matter what the economy's *initial* level of capital per effective worker, its *long-run* level of capital per effective worker will be the steady state level, k^*.

This is a more exact version of the equation—$\Delta k = sf(k) - (\delta + n + g)k$—on page 218 of the textbook. In that equation, $f(k)$ represents *output per effective worker*. For the Cobb-Douglas production function $F(K, LE) = AK^{\alpha}(LE)^{1-\alpha}$ in equation (1), it is straightforward to check that

$$
\begin{aligned}
f(k) \equiv \frac{F(K, LE)}{LE} &= \frac{AK^{\alpha}(LE)^{1-\alpha}}{LE} \\
&= AK^{\alpha}(LE)^{-\alpha} \\
&= A\frac{K^{\alpha}}{(LE)^{\alpha}} \\
&= Ak^{\alpha}. \qquad (7)
\end{aligned}
$$

Note from equation (6) that

$$
k \quad \left\{ \begin{array}{c} \text{increases over time} \\ \text{stays unchanged} \\ \text{decreases over time} \end{array} \right\} \quad \text{when} \quad sf(k) \equiv sAk^{\alpha} \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} (\delta + n + g + ng)k. \qquad (8)
$$

This equation is crucial to understanding the dynamics of the Solow Growth Model. See Figure 1 for a graphical analysis.

4 Long Run Growth: The Steady State

4.1 Calculation of Capital per Effective Worker in the Steady State

When k stays unchanged over time, we say that a *steady state* has been reached. We see from the equation above that when a steady state is reached, $sAk^{\alpha} = (\delta + n + g + ng)k$. Therefore,

$$
\begin{aligned}
\frac{k}{k^{\alpha}} &= \frac{sA}{\delta + n + g + ng} \\
k^{1-\alpha} &= \frac{sA}{\delta + n + g + ng} \\
k &= \left[\frac{sA}{\delta + n + g + ng}\right]^{\frac{1}{1-\alpha}} \equiv k^*, \qquad (9)
\end{aligned}
$$

where k^* stands for the *steady state value of capital per effective worker*, as in Figure 8–1 of the textbook.

4.2 Calculation of Capital in the Steady State, for a Given Number of Effective Workers

Now, as $k = k^*$ when the economy is in the steady state, we can write $K_t/(L_tE_t) = k^*$ or $K_t = k^* \times L_t \times E_t$. In other words, if you know that the economy is in its steady state at time t, and if you know the exact values of L_t and E_t, you can calculate the exact value of K_t. This is how I calculated the steady state value of K_t (for given values of L_t and E_t) in the "Steady State Initial Conditions" section of my Excel spreadsheet. You can check that any changes you make to the initial values of L and E in the spreadsheet will automatically change K in the "Steady State Initial Conditions" section of the spreadsheet but not in the "Arbitrary Initial Conditions" section of the spreadsheet.

4.3 Stability of the Steady State

By the way, why am I so chuffed about being able to calculate k^*? If you look carefully at equation (8)—and at Figure 1, which is a graphical view of the equation—you will see that no matter what the value of k_t may be, capital per effective worker will in the long run invariably home in on its steady state value, k^*. That's why the steady state is so significant in the Solow Growth Model.

Another way of looking at this result is that the initial values of K_t, L_t, and E_t just don't matter: we'll all end up with the same value of k no matter where we start.

4.4 Growth of Per Capita Output in the Steady State

As k is constant in the steady state, output per effective worker, which, by equation (7), is $y = k^\alpha$ is constant as well. But, as $y = Y/(LE)$, per capita output is $Y/L = y \times E$. Now, from page 25 of the textbook, we know that the growth of the product of two variables is equal to the sum of the growth rates of the two variables. Therefore, the growth rate of per capita output—which, in case you've forgotten, is what our standard of living depends on—is the growth rate of y plus the growth rate of E. The former is zero in steady state and the latter is g. Therefore, the growth rate of per capita output in the steady state is equal to g, which is the growth rate of the efficiency of labor.[6]

The initial conditions don't matter. The saving rate doesn't matter. The population growth rate doesn't matter. The only way we can speed up the rate of growth of per capita income—which, as I may have mentioned earlier, determines our standard of living—is to speed up the rate of growth of labor productivity.

5 Conclusion

With the explanations given above, the Excel spreadsheet that I use in my lectures on the Solow Growth Model should be easier to follow.

[6] See Figure 8–1 on page 219 of the textbook for more on this issue.